

ON THE POSITIVITY OF SCATTERING OPERATORS FOR POINCARÉ-EINSTEIN MANIFOLDS

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ABSTRACT. In this paper, we mainly study the scattering operators for the Poincaré-Einstein manifolds. Those operators give the fractional GJMS operators $P_{2\gamma}$ for the conformal infinity. If a Poincaré-Einstein manifolds (X^{n+1}, g_+) is locally conformally flat and there exists an representative g for the conformal infinity $(M, [g])$ such that the scalar curvature R is a positive constant and $Q_4 > 0$, then we prove that $P_{2\gamma}$ is positive for $\gamma \in (1, 2)$ and thus the first real scattering pole is less than $\frac{n}{2} - 2$.

1. INTRODUCTION

Let (X^{n+1}, g_+) be a Poincaré-Einstein manifold with smooth conformal infinity $(M, [g])$, i.e. \overline{X}^{n+1} is a smooth manifold with boundary, x is a smooth boundary defining function for $\partial X = M$ and g_+ is a smooth Riemannian metric in the interior which satisfies

$$\begin{cases} Ric_{g_+} = -ng_+ & \text{in } X, \\ x^2 g_+|_{TM} \in [g] & \text{on } M. \end{cases}$$

Here we require that $x^2 g_+$ can be $C^{k,\alpha}$ extended to the boundary with $k \geq 2[\frac{n-1}{2}] + 1$ and $0 < \alpha < 1$. Direct computation shows that all the sectional curvatures of (X^{n+1}, g_+) converge to -1 when approaching to the boundary. A standard example is the Hyperbolic space \mathbb{H}^{n+1} in the ball model:

$$X^{n+1} = \{x \in \mathbb{R}^{n+1} : |z| < 1\}, \quad g_+ = \frac{4dz^2}{(1-|z|^2)^2} = \frac{4(dr^2 + r^2 d\theta^2)}{(1-r^2)^2},$$

where (r, θ) is the polar coordinates. Take the geodesic normal defining function $x = \frac{2(1-r)}{1+r}$. Then for $x \in (0, 2)$

$$g_+ = x^{-2} \left(dx^2 + \left(1 - \frac{x^2}{4}\right)^2 d\theta^2 \right), \quad x^2 g_+|_{T\mathbb{S}^n} = d\theta^2.$$

The spectrum and resolvent for the Laplacian-Beltrami operator of (X^{n+1}, g_+) is studied by Mazzeo-Melrose in [MM], Mazzeo in [Ma] and Guillarmou [Gu]. Actually the authors dealt with more general asymptotic hyperbolic manifolds. They showed that $\text{Spec}(\Delta_+) = \sigma_{pp}(\Delta_+) \cup \sigma_{ac}(\Delta_+)$, where $\sigma_{pp}(\Delta_+)$ is the L^2 -eigenvalue set and $\sigma_{ac}(\Delta_+)$ is the absolute spectrum, and

$$\sigma_{pp}(\Delta_+) \subset \left(0, \frac{n^2}{4}\right), \quad \sigma_{ac}(\Delta_+) = \left[\frac{n^2}{4}, +\infty\right).$$

For $s \in \mathbb{C}$, $\text{Re}(s) > \frac{n}{2}$, $s(n-s) \notin \sigma_{pp}(\Delta_+)$, the resolvent $R(s) = (\Delta_+ - s(n-s))^{-1}$ defines a bounded map

$$R(s) : L^2(dV_{g_+}) \longrightarrow L^2(dV_{g_+}).$$

Moreover $R(s)$ can be meromorphically extended to $\mathbb{C} \setminus \{\frac{n-1}{2} - K - \mathbb{N}_0\}$. Here K is an integer defined by that $2K$ is the even order of g_+ in its asymptotic expansion near the boundary. For Poincaré-Einstein metric g_+ , $K \geq \frac{n-1}{2}$ for n odd and $K \geq \frac{n-2}{2}$ for n even, according to the regularity result

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given in [CDLS]. In particular, for Hyperbolic space \mathbb{H}^{n+1} , $K = +\infty$. The L^2 -eigenvalues can be estimated under certain geometric assumptions. For example, in [Le], Lee showed if (X^{n+1}, g_+) is Poincaré-Einstein and its conformal infinity is of nonnegative Yamabe type, then $\sigma_{pp}(\Delta_+) = \emptyset$.

The scattering operators associate to (X^{n+1}, g_+) are define in the following way. Consider

$$(\Delta_+ - s(n-s))u = 0, \quad x^{s-n}u|_M = f \in C^\infty(M).$$

If $\operatorname{Re}(s) > \frac{n}{2}$, $s(n-s) \notin \sigma_{pp}(\Delta_+)$, $2s-n \notin \mathbb{N}$, then

$$u = x^{n-s}F + x^sG, \quad F, G \in C^{k,\alpha}(\overline{X}), \quad F|_M = f.$$

We define the scattering operator $S(s)$ by

$$S(s) : C^\infty(M) \longrightarrow C^\infty(M), \quad S(s)f = G|_M.$$

Here $S(s)$ is a one parameter family of conformally invariant elliptic pseudo-differential operators or order $2s-n$, which can be meromorphically extended to $\mathbb{C} \setminus \{\frac{n-1}{2} - K - \mathbb{N}_0\}$ with K the same as before. If $s_0 > \frac{n}{2}$ is a pole satisfying $2s_0 - n \in \mathbb{N}$, $s_0(n-s_0) \notin \sigma_{pp}(\Delta_+)$, then the order of this pole is at most 1 and the residue is a differential operator on M . In particular, if (X^{n+1}, g_+) is a Poincaré-Einstein manifold and $\frac{n^2}{4} - k^2 \notin \sigma_{pp}(\Delta_+)$ for $k \leq \min\{\frac{n}{2}, K\}$, then

$$\operatorname{Res}_{s=s_0} S(s) = c_k P_{2k}, \quad c_k = \frac{(-1)^{k-1}}{2^{2k} k! (k-1)!}.$$

Here P_{2k} is the GJMS operator of order $2k$ on (M, g) with $g = x^2 g_+|_{TM}$. In particular, P_2 is the conformal Laplacian and P_4 is the Paneitz operator on (M, g) . See [JS], [GZ] for more details.

For simplicity, we define the renormalised scattering operators by

$$P_{2\gamma} = d_\gamma S\left(\frac{n}{2} + \gamma\right), \quad d_\gamma = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}.$$

While γ is not an integer, $P_{2\gamma}$ is also called the *fractional GJMS operators*. Similarly, the fractional Q-curvature is defined by

$$Q_{2\gamma} = \frac{2}{n-2\gamma} P_{2\gamma} 1.$$

From the definition of $P_{2\gamma}$, when γ is not an integer, $P_{2\gamma}$ should depend on the interior metric (X^{n+1}, g_+) , not only on $(M, [g])$. A special case is the Hyperbolic space \mathbb{H}^{n+1} , which has conformal infinity $(\mathbb{S}^n, [g_c])$ where $g_c = d\theta^2$ is the canonical sphere metric. In this case, the rigidity theorems given in [ST] [DJ] and [LQS] tell us if (X^{n+1}, g_+) is Poincaré-Einstein with conformal infinity $(\mathbb{S}^n, [g_c])$, then (X^{n+1}, g_+) must be the Hyperbolic space \mathbb{H}^{n+1} . So the fractional GJMS operators are uniquely defined, which are given in the following:

$$P_{2\gamma}^{g_c} = \frac{\Gamma(B + \frac{1}{2} + \gamma)}{\Gamma(B + \frac{1}{2} - \gamma)}, \quad \text{where } B = \sqrt{\Delta_{g_c} + \left(\frac{n-1}{2}\right)^2}.$$

Similarly, the fractional Q-curvature can be computed explicitly:

$$Q_{2\gamma}^{g_c} = \frac{2}{n-2\gamma} \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\frac{n}{2} - \gamma)}.$$

Using the fractional GJMS operators, we can define the fractional Yamabe Invariant by

$$\begin{aligned} Y_\gamma(M, [g]) &= \inf_{f \in C^\infty(M)} \frac{\int_M f P_{2\gamma} f \, d\operatorname{vol}_g}{\left(\int_M |f|^{\frac{2n}{n-2\gamma}} \, d\operatorname{vol}_g\right)^{\frac{n-2\gamma}{n}}} \\ &= \inf_{\hat{g} \in [g]} \frac{\frac{n-2\gamma}{2} \int_M Q_{2\gamma}^{\hat{g}} \, d\operatorname{vol}_{\hat{g}}}{\left(\int_M d\operatorname{vol}_{\hat{g}}\right)^{\frac{n-2\gamma}{2}}}. \end{aligned}$$

While $\gamma = 1$, this is the classical Yamabe invariant.

We are mainly interested in the positivity of these renormalised scattering operators. For $\gamma \in (0, 1)$ the positivity of $P_{2\gamma}$ was studied by Guillarmou-Qing in [GQ].

Theorem 1 (Guillarmou-Qing). *Suppose (X^{n+1}, g_+) ($n \geq 3$) is a Poincaré-Einstein manifold with conformal infinity $(M, [g])$. Fix a representative g for the conformal infinity and assume the scalar curvature R is positive on (M, g) . Then for $\gamma \in (0, 1)$,*

- (a) $Q_{2\gamma} > 0$ on M ;
- (b) *The first eigenvalue of $P_{2\gamma}$ is positive;*
- (c) *The Green function of $P_{2\gamma}$ is positive;*
- (d) *The first eigenspace of $P_{2\gamma}$ is spanned by a positive function.*

Based on the positivity results and the identity $S(n-s)S(s) = \text{Id}$, the authors also showed that

Theorem 2 (Guillarmou-Qing). *Suppose (X^{n+1}, g_+) ($n \geq 3$) is a Poincaré-Einstein manifold with conformal infinity $(M, [g])$. Then the Yamabe invariant $\mathcal{Y}_1(M, [g])$ is positive if and only if the first scattering pole is less than $\frac{n}{2} - 1$*

The first scattering pole has more interesting interpretation in the case of $X = \Gamma \backslash \mathbb{H}^{n+1}$, while Γ is a convex co-compact group without torsion of orientation preserving isometries of \mathbb{H}^{n+1} . In this case, the conformal infinity is locally conformally flat and given by the quotient $M = \Gamma \backslash \Omega(\Gamma)$ where $\Omega(\Gamma) \subset \mathbb{S}^n$ is the domain of discontinuity of Γ . In [Pe], Perry proved that the latest real scattering pole is given by the Poincaré exponent; and Sullivan [Su] and Patterson [Pa] showed that the Poincaré exponent of the group Γ is equal to the Hausdorff dimension δ_Γ of the limit set $\Lambda(\Gamma) = \mathbb{S}^n \setminus \Omega(\Gamma)$. Due to the work of Schoen-Yau [SY] and Nayatani [Na], δ_Γ is less than $\frac{n}{2} - 1$ if and only if the conformal infinity is of positive Yamabe Type.

In this paper, we mainly study the scattering operators of order between 2 and 4, and prove the following theorem.

Theorem 3. *Suppose (X^{n+1}, g_+) ($n \geq 5$) is a locally conformally flat Poincaré-Einstein manifold with conformal infinity $(M, [g])$ and fix a representative g for the conformal infinity. Assume the scalar curvature R is a positive constant and $Q_4 \geq 0$ on (M, g) . Then for $\gamma \in (1, 2)$,*

- (a) $Q_{2\gamma} > 0$ on M ;
- (b) *The first eigenvalue of $P_{2\gamma}$ is positive;*
- (c) *$P_{2\gamma}$ satisfies the strong maximum principle, i.e. for $f \in C^\infty(M)$, $P_{2\gamma}f \geq 0$ implies $f > 0$ or $f \equiv 0$;*
- (d) *The Green function of $P_{2\gamma}$ is positive;*
- (e) *The first eigenspace of $P_{2\gamma}$ is spanned by a positive function.*

In this case the first scattering pole $s_0 \leq \frac{n}{2} - 2$. Furthermore, if $Q_4(p) > 0$ at some point $p \in M$, then the first scattering pole $s_0 < \frac{n}{2} - 2$.

For the special case $X = \Gamma \backslash \mathbb{H}^{n+1}$ with conformal infinity $M = \Gamma \backslash \Omega(\Gamma)$, Theorem 3 implies that if there exists a representative g such that the scalar curvature R is a positive constant and $Q_4 \geq 0$, then the Hausdorff dimension δ_Γ of the limit set is less than $\frac{n}{2} - 2$. A similar result is given by Zhang [Zh] that if $R > 0$ and $Q_{2\gamma} > 0$ for some $\gamma \in (1, 2)$, then $\delta_\Gamma < \frac{n}{2} - \gamma$.

At last, we want to point out that the condition " R is a positive constant" is a technical requirement. We expect to replace it by $R > 0$ in the future. Based on the work of Gursky-Lin [GL], we also make an conjecture as follows:

Conjecture 1. *Suppose (X^{n+1}, g_+) ($n \geq 5$) is a Poincaré-Einstein manifold with conformal infinity $(M, [g])$. If the conformal infinity satisfies $\mathcal{Y}_1(M, [g]) > 0, \mathcal{Y}_2(M, [g]) > 0$, then the first scattering pole is less than $n/2 - 2$.*

2. ASYMPTOTIC COMPUTATIONS

Suppose (X^{n+1}, g_+) is a Poincaré-Einstein manifold with conformal infinity $(M, [g])$ and $n \geq 5$. In this section we do some asymptotic computations for the Poincaré-Einstein manifolds. Fix a representative g for the conformal infinity and choose x to be the geodesic normal defining function, i.e. $|dx|_{x^2 g_+}^2 = 1$ in a neighbourhood of the boundary and $x^2 g|_{TM} = g$. Then due to [CDLS], g_+ has a Taylor expansion near the boundary, i.e.

$$g_+ = \frac{1}{x^2} (dx^2 + g_0 + x^2 g_2 + x^4 g_4 + O(x^5)), \quad \text{where}$$

$$g_0 = g, \quad [g_2]_{ij} = -A_{ij}, \quad [g_4]_{ij} = \frac{1}{4(n-4)} (-B_{ij} + (n-4)A_i^k A_{jk}).$$

Here A is the Schouten tensor on (M, g) :

$$A_{ij} = \frac{1}{n-2} (R_{ij} - Jg_{ij}), \quad J = \frac{R}{2(n-1)} = Q_2;$$

and

$$B_{ij} = C_{ijk},{}^k - A^{kl}W_{kijl}; \quad C_{ijk} = A_{ij,k} - A_{ik,j}.$$

In this paper, we use \cdot to denote the covariant derivatives w.r.t. boundary metric g and \cdot to denote the covariant derivatives w.r.t. interior metric g_+ . Then the Laplacian of g_+ , denoted by Δ_+ , also has an expansion near the boundary, i.e.

$$\Delta_+ = -(x\partial_x)^2 + nx\partial_x + x^2 L_2 + x^4 L_4 + O(x^5)$$

where

$$L_2 = \Delta_g + Jx\partial_x,$$

$$L_4 = \delta_g Ad + \frac{1}{2}(-\delta_g Jd + J\Delta_g) + \frac{1}{2}|A|_g^2 x\partial_x$$

We also denote

$$L_2(s) = \Delta_g + sJ,$$

$$L_4(s) = \delta_g Ad + \frac{1}{2}(-\delta_g Jd + J\Delta_g) + \frac{1}{2}s|A|_g^2$$

For $f \in C^\infty(M)$ and $\text{Re}(s) > \frac{n}{2}$, $2s - n \notin \mathbb{N}$ and $s(n-s) \notin \sigma_{pp}(\Delta_+)$, consider the equation

$$\Delta_{g_+} u - s(n-s)u = 0, \quad x^{s-n}u|_M = f.$$

Then $u = x^{n-s}F + x^sG$ with $F, G \in C^\infty(\overline{X})$. Moreover, F has asymptotical expansion

$$F = f - \frac{x^2}{2(2s-n-2)} T_2(n-s)f + \frac{x^4}{8(2s-n-2)(2s-n-4)} T_4(n-s)f + O(x^5),$$

and $G = S(s)f + O(x^2)$. Here

$$T_2(n-s) = L_2(n-s),$$

$$T_4(n-s) = L_2(n-s+2)L_2(n-s) - 2(2s-n-2)L_4(n-s).$$

By direct computation,

$$T_2(n-s) = P_2 + \frac{n+2-2s}{2}J,$$

$$T_4(n-s) = P_4 + \frac{n+4-2s}{2} (2J\Delta_g + 4\delta_g Ad + Q_4 + (n-s)(J^2 + 2|A|_g^2)).$$

If $s = \frac{n}{2} + 1$, then

$$T_2\left(\frac{n}{2} - 1\right) = P_2 = \Delta_g + \frac{n-2}{2}J.$$

If $s = \frac{n}{2} + 2$, then

$$T_4 \left(\frac{n}{2} - 2 \right) = P_4 = \Delta_g^2 + \delta_g((n-2)J - 4A) d + \frac{n-4}{2} Q_4.$$

3. THE POSITIVITY OF $P_{2\gamma}$ FOR $\gamma \in (1, 2)$

We mainly prove Theorem 3 in this section. The positivity of fractional GJMS operators is studied carefully in Section 7 of [CC]. Combine their results with the spectrum theorem in [Le], we first know that

Proposition 1 (Case-Chang, Lee). *Let (X^{n+1}, g_+) ($n \geq 4$) be a Poincaré-Einstein manifold with conformal infinity $(M, [g])$. Fix a representative g for the conformal infinity. Assume the scalar curvature $R > 0$ and $Q_{2\gamma} > 0$ for some $\gamma \in (1, 2)$. Then*

- (a) *There is no L^2 -eigenvalue for Δ_+ , i.e. $\text{spec}(\Delta_+) = [n^2/4, \infty)$;*
- (b) *The first eigenvalue of $P_{2\gamma}$ satisfies $\lambda_1(P_{2\gamma}) \geq \min_M Q_{2\gamma} > 0$;*
- (c) *$P_{2\gamma}$ satisfies strong maximum principle: if $P_{2\gamma}f \geq 0$ for $f \in C^\infty(M)$, then $f > 0$ or $f \equiv 0$;*
- (d) *The Green's function of $P_{2\gamma}$ is positive.*

Therefore, to prove Theorem 3, we only need to prove part (a) and part (e). Here we work out a comparison theorem similar as Guillarmou-Qing did in [GQ].

For $\gamma \in (1, 2)$, set $s = \frac{n}{2} + \gamma$ and let u solves the following equation:

$$(1) \quad \Delta_+ u - s(n-s)u = 0, \quad x^{s-n}u|_M = 1.$$

Then u is positive on X and near the boundary u has an asymptotic expansion as follows:

$$u = x^{n-s} (1 + x^2 u_2 + x^{2\gamma} u_{2\gamma} + x^4 u_4 + O(x^5)),$$

where

$$\begin{aligned} u_{2\gamma} &= S(s)1 = 2^{-2\gamma} \frac{\Gamma(-\gamma)}{\Gamma(\gamma)} \frac{n-2\gamma}{2} Q_{2\gamma}, \\ u_2 &= -\frac{(n-s)}{2(2s-n-2)} J < 0, \\ u_4 &= \frac{(n-s)}{8(2s-n-2)(2s-n-4)} \left(Q_4 + \frac{n+4-2s}{2} (J^2 + 2|A|_g^2) \right) < 0. \end{aligned}$$

We also define a test function ψ by

$$(2) \quad \psi = v^{\frac{n-s}{n-\lambda}} \quad \text{where} \quad \begin{cases} \Delta_+ v - \lambda(n-\lambda)v = w, & x^{\lambda-n}v|_M = 1; \\ \Delta_+ w - (\lambda-2)(n-\lambda+2)w = 0, & x^{\lambda+2-n}w|_M = w_0. \end{cases}$$

Here $\lambda > \max\{s+2, n\}$ will be fixed later and

$$w_0 = \frac{-2(\lambda-s)(n-\lambda)}{2s-n-2} J > 0.$$

So by maximum principle, $w > 0$, $v > 0$ and hence $\psi > 0$ on X .

Lemma 1. *Near the boundary M , ψ has an asymptotic expansion*

$$\psi = x^{n-s} (1 + \psi_2 x^2 + \psi_4 x^4 + O(x^5)),$$

satisfying

$$\psi_2 = u_2.$$

Assume $J > 0$ and $Q_4 \geq 0$, then

$$\psi_4 > u_4.$$

Proof. Here v, w have asymptotic expansions as follows

$$\begin{aligned} v &= x^{n-\lambda} (1 + v_2 x^2 + v_4 x^4 + O(x^5)), \\ w &= x^{n-\lambda+2} (w_0 + w_2 x^2 + O(x^3)). \end{aligned}$$

By direct computation

$$\begin{aligned} w_2 &= \frac{(\lambda-s)(n-\lambda)}{(2\lambda-n-6)(2s-n-2)} (\triangle_g J + (n-\lambda+2)J^2), \\ v_2 &= -\frac{(n-\lambda)}{2(2s-n-2)} J, \\ v_4 &= \frac{1}{4(2\lambda-n-4)} \left[\left(\frac{\lambda-s}{2\lambda-n-6} + \frac{1}{2} \right) \frac{n-\lambda}{2s-n-2} (\triangle_g J + (n-\lambda+2)J^2) - \frac{n-\lambda}{2} |A|^2 \right]. \end{aligned}$$

This implies

$$\begin{aligned} \psi_2 &= -\frac{(n-s)}{2(2s-n-2)} J, \\ \psi_4 &= \frac{n-s}{n-\lambda} v_4 + \frac{(n-s)(\lambda-s)}{2(n-\lambda)^2} v_2^2. \end{aligned}$$

So $\psi_2 = u_2$ and

$$\begin{aligned} \psi_4 - u_4 &= \frac{(n-s)(\lambda-s)}{4} \left[\frac{Q_4}{(2\lambda-n-4)(2s-n-2)} \left(\frac{1}{2\lambda-n-6} - \frac{1}{2s-n-4} \right) \right] \\ &\quad + \frac{(n-s)(\lambda-s)}{4(2s-n-2)^2(2\lambda-n-6)} [(2s-n-2)|A|_g^2 + (\lambda-s-2)J^2]. \end{aligned}$$

Since $\lambda > s+2$, $\lambda > n > s$ and $J > 0$, $Q_4 \geq 0$, we have $\psi_4 - u_4 > 0$. □

For the global behaviour of ψ , a direct computation gives

$$\begin{aligned} \frac{\triangle_+ \psi}{\psi} - s(n-s) &= (n-s) \left[(\lambda-s) \left(1 - \frac{1}{(n-\lambda)^2} v^{-2} |\nabla v|_+^2 \right) + \frac{1}{n-\lambda} v^{-1} w \right] \\ &= (n-s) K x^4 + O(x^5), \end{aligned}$$

where

$$\begin{aligned} K &= -(\lambda-s) \left[\frac{1}{(2s-n-2)^2} J^2 - \frac{1}{2\lambda-n-4} |A|_g^2 \right. \\ &\quad \left. + \frac{2(\lambda-s-1)}{(2\lambda-n-4)(2\lambda-n-6)(2s-n-2)} (\triangle_g J + (n-\lambda+2)J^2) \right] \\ &= -(\lambda-s) \left[\frac{(n+4-2s)}{(2\lambda-n-6)(2s-n-2)} ((\lambda-s-2)J^2 + |A|_g^2) \right. \\ &\quad \left. + \frac{2(\lambda-s-1)}{(2\lambda-n-4)(2\lambda-n-6)(2s-n-2)} Q_4 \right]. \end{aligned}$$

Lemma 2. Assume $\lambda > s+2$, $\lambda > n > s$ and $J > 0$, $Q_4 \geq 0$ on M , then $K < 0$.

Next take $\lambda = n+2$ in (2) to simplify the computations. In this case, $v^{-2} = O(x^4)$. Denote

$$\frac{\triangle_+ \psi}{\psi} - s(n-s) = (n-s) v^{-2} I,$$

where

$$(3) \quad I = (n+2-s) \left(v^2 - \frac{1}{4} |\nabla v|_+^2 \right) - \frac{1}{2} (vw).$$

Obviously $I|_M = K$.

Lemma 3. For I defined in (3) we have

$$\begin{aligned} \Delta_+ I = & (n+2-s) \left(\frac{1}{2} |\nabla^2 v|_+^2 + \frac{n}{2} |\nabla v|_+^2 - 4(n+2)v^2 \right) \\ & - \frac{(n-s)}{2} \langle \nabla v, \nabla w \rangle_+ + [3(n+2) - 2s]vw - \frac{1}{2}w^2. \end{aligned}$$

Proof. Here on X^{n+1} , $R_{ij}^+ = -n[g_+]_{ij}$ and v, w satisfy

$$\begin{aligned} v_{:k}^k &= 2(n+2)v - w, \\ w_{:k}^k &= 0. \end{aligned}$$

Here the $:$ denotes the covariant derivative w.r.t. g_+ . Hence

$$\begin{aligned} v_{:ik}^k &= v_{:ki}^k + R_{ik}^+ v^k = (n+4)v_i - w_i, \\ w_{:ik}^k &= w_{:ki}^k + R_{ik}^+ w^k = -nw_i. \end{aligned}$$

Direct computation shows that

$$\begin{aligned} (v^2)_{:k}^k &= 2(v^k v_k + v v_{:k}^k) = 2(|\nabla v|_+^2 + v[2(n+2)v - w]), \\ (|\nabla v|_+^2)_{:k}^k &= 2(v_{:ik}^k v_{:ik}^k + v^i v_{:ik}^k) = 2(|\nabla^2 v|_+^2 + (n+4)|\nabla v|_+^2 - \langle \nabla v, \nabla w \rangle_+), \\ (vw)_{:k}^k &= v_{:k}^k w + 2v_k w^k + v w_{:k}^k = 2\langle \nabla v, \nabla w \rangle_+ + [2(n+2)v - w]w. \end{aligned}$$

Hence we get the formula for $\Delta_+ I = -I_{:k}^k$. □

Denote by

$$(4) \quad II = \Delta_+ I.$$

Using the asymptotical expansion of Δ_+ , it is obvious that $II|_M = 0$.

Lemma 4. For II defined in (4),

$$\begin{aligned} \Delta_+ II + 2(n+2)II = & (n+2-s) \left(-|\nabla^3 v|_+^2 + (6n+16)|\nabla v|_+^2 + 2W_{ikjl}^+ v^{ij} v^{kl} \right) \\ & + 2(n+1-s) \langle \nabla^2 v, \nabla^2 w \rangle_+ - 4(n+3-s) \langle \nabla v, \nabla w \rangle_+ + \frac{n-s-2}{2} |\nabla w|_+^2. \end{aligned}$$

Proof. Let R_{imjk}^+ be the Riemannian curvature tensor of (X^{n+1}, g_+) . Then since $R_{ij}^+ = -n[g_+]_{ij}$,

$$R_{imjk}^+ = W_{imjk}^+ - [g_+]_{ij}[g_+]_{mk} + [g_+]_{ik}[g_+]_{mj},$$

Moreover, $R_{ij:k}^+ = 0$ and $R_{imjk:}^k = 0$. The later is because of Bianchi identity

$$R_{jkim:l}^+ + R_{jkml:i}^+ + R_{jkli:m}^+ = 0.$$

Therefore,

$$\begin{aligned} v_{:ij}^{ij} v_{:ijkl} &= v_{:ij}^{ij} (v_{:ikj} - R_{imjk}^+ v^m)_{:l} \\ &= v_{:ij}^{ij} (v_{:kijl} - R_{imjk}^+ v_{:l}^m - R_{imjk:l}^+ v^m) \\ &= v_{:ij}^{ij} (v_{:kilj} - R_{kmjl}^+ v_{:i}^m - R_{imjl}^+ v_{:k}^m - R_{imjk}^+ v_{:l}^m - R_{imjk:l}^+ v^m) \\ &= v_{:ij}^{ij} [(v_{:kli} - R_{kmil}^+ v^m)_{:j} - R_{kmjl}^+ v_{:i}^m - R_{imjl}^+ v_{:k}^m - R_{imjk}^+ v_{:l}^m - R_{imjk:l}^+ v^m] \\ &= v_{:ij}^{ij} (v_{:klij} - R_{kmil}^+ v_{:j}^m - R_{kmil:j}^+ v^m - R_{kmjl}^+ v_{:i}^m - R_{imjl}^+ v_{:k}^m - R_{imjk}^+ v_{:l}^m - R_{imjk:l}^+ v^m). \end{aligned}$$

So

$$\begin{aligned} v_{:ij}^{ij} v_{:ijkl} &= v_{:ij}^{ij} (v_{:kij} + R_{mi}^+ v_{:j}^m + R_{mi:j}^+ v^m + R_{mj}^+ v_{:i}^m - 2R_{imjk}^+ v_{:}^{mk} - R_{imjk:}^k v^m) \\ &= v_{:ij}^{ij} (v_{:kij} - 2n v_{:ij} + 2v_{:k}^k [g_+]_{ij} - 2v_{:ij} - 2W_{imjk}^+ v_{:}^{mk}). \end{aligned}$$

Since $v_{:k}^k = 2(n+2)v - w$, we have

$$v_{:ij} v_{:ijk}^k = 2|\nabla^2 v|_+^2 - \langle \nabla^2 v, \nabla^2 w \rangle_+ + 2[2(n+2)v - w]^2 - 2W_{imjk}^+ v_{:ij} v_{:mk}.$$

Hence

$$(|\nabla^2 v|_+^2)_{:k}^k = 2(|\nabla^3 v|_+^2 + 2|\nabla^2 v|_+^2 - \langle \nabla^2 v, \nabla^2 w \rangle_+ + 2[2(n+2)v - w]^2 - 2W_{imjk}^+ v_{:mk}).$$

Moreover,

$$\begin{aligned} (\langle \nabla v, \nabla w \rangle_+)_{:k}^k &= v_{:ik}^k w^i + 2v_{:ik} w_{:ik}^k + v^i w_{:ik}^k = 2\langle \nabla^2 v, \nabla^2 w \rangle_+ + 4\langle \nabla v, \nabla w \rangle_+ - |\nabla w|_+^2, \\ (w^2)_{:k}^k &= 2w_k w^k + 2w w_{:k}^k = 2|\nabla w|_+^2. \end{aligned}$$

And we get the formula for $\Delta_+ II = -II_{:k}^k$. □

Lemma 5. *Assume the (X^{n+1}, g_+) is locally conformally flat; J is a positive constant and $Q_4 \geq 0$ on M . Then the test function ψ satisfies*

$$\Delta_+ \psi - s(n-s)\psi < 0$$

all over X .

Proof. Since J is a positive constant, $w = w_0$ is also a positive constant. By Lemma 2, $I|_M = K < 0$ and by Lemma 3,

$$II = \Delta_+ I = (n+2-s) \left(\frac{1}{2} |\nabla^2 v|_+^2 + \frac{n}{2} |\nabla v|_+^2 - 4(n+2)v^2 \right) + [3(n+2) - 2s]vw - \frac{1}{2}w^2.$$

Since (X^{n+1}, g_+) is locally conformally flat, the Weyl tensor $W^+ = 0$. By Lemma 4,

$$\Delta_+ II + 2(n+2)II = (n+2-s) \left(-|\nabla^3 v|_+^2 + (6n+16)|\nabla v|_+^2 \right) = -(n+2-s)|V|_+^2 \leq 0,$$

where

$$V_{ijk} = v_{:ijk} - v_i[g_+]_{jk} - v_j[g_+]_{ik} - 2v_k[g_+]_{ij}.$$

Notice that $II|_M = 0$. Therefore, by maximum principle

$$II = \Delta_+ I \leq 0,$$

all over X , which together with $I|_M = K < 0$ implies that

$$I < 0$$

all over X . □

Proposition 2. *Let (X^{n+1}, g_+) ($n \geq 5$) be a locally conformally flat Poincaré-Einstein manifold with conformal infinity $(M, [g])$. Fix a representative g for the conformal infinity. Assume the scalar curvature R is a positive constant and $Q_4 \geq 0$ on M . Then for all $\gamma \in (1, 2)$, $Q_{2\gamma} > 0$.*

Proof. Similar as Guillarmou-Qing did in [GQ], we compare the two functions u and ψ , which are defined in (1) and (2) with $\lambda = n+2$. First u/ψ satisfies the equation:

$$\Delta_+ \left(\frac{u}{\psi} \right) = \left(s(n-s) - \frac{\Delta_+ \psi}{\psi} \right) \frac{u}{\psi} + 2\nabla \left(\frac{u}{\psi} \right) \frac{\nabla \psi}{\psi}, \quad \left(\frac{u}{\psi} \right)_M = 1.$$

Notice that $u/\psi > 0$. Applying maximum principle, we show that u/ψ can not attain an interior positive minimum. Hence $u/\psi \geq 1$, i.e. $u \geq \psi$. Near the boundary, this means

$$1 + x^2 u_2 + x^{2\gamma} u_{2\gamma} + x^4 u_4 + O(x^5) \geq 1 + x^2 \psi_2 + x^4 \psi_4 + O(x^5)$$

Since $\psi_2 = u_2$ and $\psi_4 > u_4$, we have $u_{2\gamma} > 0$. Hence $Q_{2\gamma} > 0$ on M . □

Proof of Theorem 3: Part (a) is proved in Proposition 2. Part (b) (c) (d) are from Proposition 2 and Proposition 1. Part (e) is proved by the same proof for Proposition 4.2 in [GQ]. Recall the positive results in [GQ]. Then in our setting, $P_{2\gamma}$ is positive for all $\gamma \in [0, 2)$ and hence the first scattering pole $s_0 \leq \frac{n}{2}$. Furthermore, if $Q_4(p) > 0$ at some point p , then P_4 is also positive. Therefore the first scattering pole $s_0 < \frac{n}{2}$.

REFERENCES

- [Be] Beckner, William: *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. **138**(2) (1993), 213-243.
- [CC] Chang, Sung-Yung Alice; Case, Jeffrey: *On fractional GJMS operators*, Pure Appl. Math., to appear. DOI: 10.1002/cpa.21564.
- [CDLS] Chruściel, Piotr T.; Delay, Erwann; Lee, John M.; Skinner, Dale N.: *Boundary regularity of conformal compact einstein metrics*, J. Diff. Geom. **69** (2005), 111-136.
- [CT1] Cotsiolis, A.; Tavoularis, N. C.: *Sharp Sobolev type inequalities for higher fractional derivatives*, C. R. Acad. Sci. Paris. Ser. I **335** (2002) 801-804.
- [CT2] Cotsiolis, A.; Tavoularis, N. C.: *Best constants for Sobolev inequalities for higher order fractional derivatives*, J. Math. Anal. Appl. **295** (2004) 225-236.
- [DJ] Dutta, Satyaki; Javaheri, Mohammad: *Rigidity of conformally compact manifold with the round sphere as the conformal infinity*, Adv. Math. **224** (2010) 525-538.
- [GL] Gursky, Matthew J.; Lin, Yueh-Ju: *The Q-curvature and manifolds with positive Yamabe invariant*, arXiv: 1502.01050.
- [Gu] Guillarmou, Colin: *meromorphic properties of the resolvent for asymptotically hyperbolic manifolds*, Duke Math. J. **129** (2005), No. 1, 1-37.
- [GQ] González, María del Mar; Qing, Jie: *Fractional conformal Laplacians and fractional Yamabe problems*, APDE, Vol. **6** (2013), No. 7, 1535-1576.
- [GZ] Graham, C. Robin; Zoworski, Maciej: *Scattering matrix in conformal geometry*, Invent. Math. **152** (2003), 89-118.
- [JS] Joshi, M; Sá Barreto: *Inverse scattering on asymptotically hyperbolic manifolds*, Acta. Math. **184** (2000), 41-86.
- [Le] Lee, John M.: *The spectrum of an asymptotically hyperbolic Einstein manifold*, Comm. Anal. Geom. **3** (1995), No. 1-2, 253-271.
- [LQS] Li, Gang; Qing, Jie; Shi, Yuguang: *Gap phenomena and curvature estimates for conformally compact Einstein manifolds*, to appear in Transactions of the American Mathematical Society, arXiv: 1410.6402.
- [MM] Mazzeo, Rafe; Melrose, Richard B.: *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), 260-310.
- [Ma] Mazzeo, Rafe: *Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds*, American Journal of Mathematics **113** (1991), No. 1, 25-45.
- [Na] Nayatani, Sh: *Patterson-Sullivan measure and conformally flat metrics*, Mathematische Zeitschrift **225**, no. 1 (1997): 115-131.
- [Pa] Patterson, S. J.: *The limit set of a Fuchsian group*, Acta Mathematica **136**, no. 3-4 (1976): 241-273.
- [Pe] Perry, P. A.: *The Laplace operator on a hyperbolic manifold. II. Eisenstein series and the scattering matrix*, Journal für die Reine und Angewandte Mathematik **398** (1989): 67-91.
- [ST] Shi, Yuguang; Tian, Gang: *Rigidity of asymptotically hyperbolic manifolds*, Commun. Math. Phys. **259** (2005), 545-559.
- [SY] Schoen, R.; Yau, S. T.: *Conformally flat manifolds, Kleinian groups and scalar curvature*, Inventiones Mathematicae **92** (1988): 47-71.
- [Su] Sullivan, D: *The density at infinity of a discrete group of hyperbolic motions*, Publications Mathématiques de l'Institut des Hautes études Scientifiques **50** (1979): 171-202.
- [Zh] Zhang, Ruobing: *Nonlocal curvature and topology of locally conformally flat manifolds*, arXiv: 1510.00957.

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